J. DIFFERENTIAL GEOMETRY
8 (1973) 75-84

SECOND ORDER CONNECTIONS. II

ROBERT H. BOWMAN

1. Introduction

The purpose of this paper is the development of certain implications of the second order connection, introduced previously by the present writer [1]. If M is an n-dimensional C^∞ manifold, we show that a linear second order connection on M determines a "covariant derivative" ∇' on TM, which satisfies the usual conditions over the ring $\mathfrak{F}'(TM)$, the vertical lift of the ring $\mathfrak{F}(M)$ of C^∞ functions on M. Using the properties of ∇', we obtain equations analogous to those of Gauss and Weingarten, and an analog of the second fundamental form.

If A, B, $C \in \mathfrak{X}'(TM)$, the module of C^∞ vector fields on TM over the ring $\mathfrak{F}'(TM)$, then we obtain the maps Tor (A, B) and $R(A, B)C$ which are $\mathfrak{F}'(TM)$ multilinear analogs of the torsion and curvature tensors. From the components of R we obtain equations analogous to those of Gauss and Codazzi, as well as an additional equation which defines a "vertical curvature tensor" on M. Finally, we obtain an invariant which we call the second order curvature of M; this yields as a special case the usual (first order) curvature of M.

2. Preliminary remarks

In this section we will briefly outline the main results of [1] utilized in the main body of this paper. The notation employed is essentially that of [1] and [2], with the summation convention employed on lower case Latin indices.

A second order connection on M is a connection on the bundle ${}^2_0\Pi : {}^2M \to M$ which naturally induces a (first order) connection on M. If ${}^1_0\Pi_*$ is the tangent map of ${}^1_0\Pi : TM \to M$, and $\tilde{D}$ is the connection map of the induced connection, then TTM and consequently ${}^2M$ may be given a vector bundle structure over M, such that if HTM and VTM are the horizontal and vertical subbundles of TTM determined by the vector bundle structure, then

$$ {}^1_0\Pi_* : HTM_p \to TM_{{}^1_0\Pi(p)} , \qquad \tilde{D} : VTM_p \to TM_{{}^1_0\Pi(p)} $$

are isomorphisms at each $p \in TM$.

Given a coordinate chart (U, ϕ) of M there are determined two sets of bases, relative to the induced coordinates $x^{01}, \cdots, x^{0n}; x^{11}, \cdots, x^{1n}$ on ${}^1_0\Pi^{-1}(U)$,

Communicated by K. Yano, August 23, 1971. Partially supported by an Arkansas State University research grant.

$$(1) \qquad X_i^h = \partial/\partial x^{0i} - \Gamma_{ik}^j x^{1k}\partial/\partial x^{1j}\,, \qquad X_i^v = \partial/\partial x^{1i}$$

respectively spanning HTM_p and VTM_p at each $p \in {}_0^1\Pi^{-1}(U)$. Similarly, there are two sets of bases

$$X_i^h = \partial/\partial x^{0i} - \Gamma_{ii}^j\partial/\partial x^{1j}\,, \qquad X_i^v = \partial/\partial x^{1i}$$

spanning the vertical and horizontal subbundles of ${}_0^2\Pi : {}^2M \to M$ at each $x \in U$. We call these subbundles the horizontal and vertical bundles over M.

A second order connection on M determines a covariant differentiation of a section A of ${}_0^2\Pi : {}^2M \to M$ with respect to a vector field X on M. The local form of this differentiation in terms of local coordinates on M is

$$(2) \qquad D_X A = \xi^j\left(\frac{\partial A^{0i}}{\partial x^{0j}} + \Gamma_{jk}^{0i}A^{0k}\right)X_i^h + \xi^j\left(\frac{\partial A^{1i}}{\partial x^{0j}} + \Gamma_{j0k}^{1i}A^{0k} + \Gamma_{j1k}^{1i}A^{1k}\right)X_i^v\,,$$

where $X = \xi^i\partial/\partial x^{0j}$ and $A = A^{0i}X_i^h + A^{1i}X_i^v$.

3. The $\mathfrak{F}'$-derivative

Theorem 1. *A second order linear connection on M determines a C^∞ map $\nabla' : \mathfrak{X}'(TM) \times \mathfrak{X}'(TM) \to \mathfrak{X}'(TM)$ such that if $A, B, C \in \mathfrak{X}'(TM)$ and $f \in \mathfrak{F}'(TM)$, then*

1) $\nabla'_A(B + C) = \nabla'_A B + \nabla'_A C$,
2) $\nabla'_{A+B}C = \nabla'_A C + \nabla'_B C$,
3) $\nabla'_{fA}B = f\nabla'_A B$,
4) $\nabla'_A fB = (Af)B + f\nabla'_A B$.

We call ∇' an $\mathfrak{F}'$-derivative.

Proof. Form the map

$${}_0^1\Pi_* \oplus \tilde{D} : TTM \to TM \oplus TM\,.$$

Since ${}^2M \approx TM \oplus TM$, we may regard ${}_0^1\Pi_* \oplus \tilde{D}$ as a map of TTM onto ${}^2M$ which is an isomorphism of TM_p onto ${}^2M_{{}_0^1\Pi(p)}$ at each $p \in TM$. Suppose that $A, B \in \mathfrak{X}'(TM)$ and that $B^h \in \mathfrak{X}'(TM)$ is the vector field obtained by taking the horizontal component of B_p at each $p \in TM$. If $\sigma_p(t)$ is an integral curve of B^h through $p \in TM$, then ${}_0^1\Pi_* B_p^h$ is tangent to ${}_0^1\Pi \cdot \sigma_p(t)$ at ${}_0^1\Pi(p)$. Since ${}_0^1\Pi_* \oplus \tilde{D}(A)$ is a well defined section of ${}_0^2\Pi : {}^2M \to M$ along ${}_0^1\Pi \cdot \sigma_p(t)$,

$$D_{{}_0^1\Pi_* B_p^h}\, {}_0^1\Pi_* \oplus \tilde{D}(A)$$

is defined and we take

$$(3) \qquad (\nabla'_B A)_p = ({}_0^1\Pi_* \oplus \tilde{D})_p^{-1}(D_{{}_0^1\Pi_* B_p^h}\, {}_0^1\Pi_* \oplus \tilde{D}(A))\,,$$

where $({}^1_0\Pi_* \oplus \tilde{D})_p^{-1}$ denotes the inverse of the isomorphism ${}^1_0\Pi_* \oplus \tilde{D}: TM_p \to {}^2M_{{}^1_0\Pi(p)}$. That $\nabla'_B A$ is C^∞ on TM follows from the fact that $\sigma_p(t)$ is C^∞ as a function of p. Conditions 1)–4) follow from the fact that if $f' \in \mathfrak{F}'(TM)$ is the vertical lift of $f \in \mathfrak{F}(M)$ and $B \in \mathfrak{X}'(TM)$, then ${}^1_0\Pi_*(f'B)_p = f({}^1_0\Pi(p))\, {}^1_0\Pi_* B_p$ and $(Bf')_p = ({}^1_0\Pi_* B_p)f$, together with the local expression (2).

Lemma 1. *If $p \in TM_0$, then $(\nabla'_B A)$ depends only on the value of B^h at p and the values of A on TM_0.*

Proof. If $p \in TM_0$ and $B \in \mathfrak{X}'(TM)$, then the integral curve σ_p of B^h through p lies entirely in the zero section TM_0 of ${}^1_0\Pi: TM \to M$. Hence ${}^1_0\Pi_* \oplus \tilde{D}(A)$ depends only on the values of $A \mid TM_0$. That $(\nabla'_B A)_p$ depends only on the value of B^h at p follows from (3).

Since the restriction of ${}^1_0\Pi$ to TM_0 is a diffeomorphism ${}^1_0\Pi: TM_0 \to M$, the restriction of ${}^1_0\Pi_*$ to $T(TM_0)$ is an isomorphism ${}^1_0\Pi_*: T(TM_0) \to TM$. Because the second order connection is linear, the induced first order connection is also. This means that if we choose a point of M and a coordinate neighborhood containing it, then the induced local bases for HTM and $T(TM_0)$ coincide on TM_0. Thus we may identify the bundle ${}^1_0\Pi: TM \to M$ with the subbundle ${}^2_1\Pi: HTM_0 \to TM_0$.

If $X, Y \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}^v(M)$, the module of vertical vector fields on $TM_0 \approx M$, then we may utilize the above identification and the fact that Lemma 1 implies that ∇' may be restricted to $TM_0 \approx M$ to decompose $\nabla'_X Y$ and $\nabla'_X \xi$ into horizontal and vertical components. Thus using (2) we have, on $TM_0 \approx M$,

$$\nabla'_X Y = D_X Y + \alpha(X, Y)\,, \qquad \nabla'_X \xi = \nabla_X \xi\,. \tag{4}$$

Theorem 2. *If $X, Y \in \mathfrak{X}(M)$, and $\xi \in \mathfrak{X}^v(M)$, then*

1) *the horizontal component $D_X Y$ of $\nabla'_X Y$ is the usual covariant derivative of the induced connection,*

2) *the vertical component $\alpha(X, Y)$ of $\nabla'_X Y$ is bilinear over $\mathfrak{F}(M)$,*

3) *$\nabla: \mathfrak{X}(M) \times \mathfrak{X}^v(M) \to \mathfrak{X}^v(M)$ satisfies the usual conditions* (1)–4) *of ∇') of a covariant derivative over $\mathfrak{F}(M)$.*

Proof. 1) D is obtained by taking the horizontal component of the restriction of ∇' to horizontal vector fields on TM_0. The induced (first order) covariant derivative may be obtained by taking the horizontal component of the second order covariant derivative restricted to horizontal vector fields of ${}^2_0\Pi: {}^2M \to M$. Since ${}^1_0\Pi_* \oplus \tilde{D}$ may be viewed as an identification of $TTM \mid TM_0$ with ${}^2M$ we see that D is the induced (first order) covariant derivative on M.

2) That α is bilinear over $\mathfrak{F}(M)$ may be seen by noting that since

$$\nabla'_X fY - D_X fY = \alpha(X, fY)$$

and

$$f\nabla'_X Y + (Xf)Y - fD_X Y - (Xf)Y = f(\alpha(X, Y)) ,$$

we have

$$\alpha(X, fY) = f\alpha(X, Y) .$$

Similarly, we see that

$$\alpha(fX, Y) = f\alpha(X, Y) .$$

3) The fact that ∇ satisfies conditions 1)–4) over $\mathfrak{F}(M)$ follows from the fact that ∇' satisfies these conditions over $\mathfrak{F}'(TM)$, and the fact that if $f' \in \mathfrak{F}'(TM)$ then it is the vertical lift of some function $f \in \mathfrak{F}(M)$ with $f' \mid TM_0 = f$ (since $TM_0 \approx M$).

In analogy with the equations of Gauss and Weingarten we call α the second fundamental form of the second order connection, and note that ∇ represents covariant differentiation with respect to a connection in the vertical bundle over M, and that the Weingarten map vanishes identically.

If $A, B \in \mathfrak{X}'(TM)$, we define

$$\text{(5)} \qquad \operatorname{Tor}(A, B) = \nabla'_A B - \nabla'_B A - [A, B] .$$

Theorem 3. *The map* $\operatorname{Tor}: \mathfrak{X}'(TM) \times \mathfrak{X}'(TM) \to \mathfrak{X}'(TM)$ *is skew-symmetric and bilinear over* $\mathfrak{F}'(TM)$.

Since Tor is bilinear over $\mathfrak{F}'(TM)$ but not over $\mathfrak{F}(TM)$, $\operatorname{Tor}(A, B)_p$ depends in general on the behavior of A and B in a neighborhood of p; however, we may localize Tor on TM_0.

Lemma 2. *If* $A, B \in \mathfrak{X}'(TM)$ *are horizontal, and* $p \in TM_0$, *then* $[A, B]_p$ *depends only upon the values of* A *and* B *on* TM_0.

Proof. If (U, ϕ) is a coordinate chart at ${}^0_1\Pi(p)$, then $A = a^i X^h_i$, $B = b^j X^h_j$, a^i, $b^j \in \mathfrak{F}(TM)$. Thus $[A, B]_p = (a^i(X^h_i b^j)X^h_j + a^i b^j X^h_i X^h_j - b^j(X^h_j a^i)X^h_i - b^j a^i X^h_j X^h_i)_p$ Since $p \in TM_0$, $(X^h_i)_p = (\partial/\partial x^{0i})(p)$ and thus we have

$$\text{(6)} \qquad [A, B]_p = [A \mid TM_0, B \mid TM_0]_p .$$

Theorem 4. *If* $p \in TM_0$ *and* $A, B \in \mathfrak{X}'(TM)$ *are horizontal, then*

$$\operatorname{Tor}(A_p, B_p) = \operatorname{Tor}(A, B)_p .$$

Proof. If A and B are horizontal vector fields on TM, and (U, ϕ) is a coordinate chart at ${}^1_0\Pi(p)$, then $A \mid TM_0 = a^i X^h_i \mid TM_0$, $B \mid TM_0 = b^j X^h_j \mid TM_0$ where $a^i, b^j \in \mathfrak{F}(M)$. Extend these to the vector fields

$$\bar{A} = a^{i\prime} X^h_i , \qquad \bar{B} = b^{j\prime} X^h_j ,$$

where $a^{i'}$ and $b^{j'}$ are the vertical lifts of a^i and b^j respectively. For $p \in TM_0$ we have by Lemmas 1 and 2

$$\mathrm{Tor}\,(A, B)_p = \mathrm{Tor}\,(\bar{A}, \bar{B})_p = \mathrm{Tor}\,(a^{i'}X_i^h, b^{j'}X_j^h)_p = a^{i'}b^{j'}\,\mathrm{Tor}\,(X_i^h, X_j^h)_p\ ,$$

and we see that if A or B vanishes at a point $p \in TM_0$, then $\mathrm{Tor}\,(A, B)_p = 0$. Hence we may take

$$(7)\qquad \mathrm{Tor}\,(A_p, B_p) = \mathrm{Tor}\,(A, B)_p\ .$$

Remark. This implies that Tor induces a tensor on $TM_0 \approx M$, since the restriction of $\mathfrak{F}'(TM)$ to TM_0 may be identified with $\mathfrak{F}(M)$.

Theorem 5. *If ∇' is torsion free* (Tor $\equiv 0$ *on* TM_0), *then the induced (first order) covariant derivative is torsion free and α is symmetric.*

Proof. Suppose that $X, Y \in \mathfrak{X}(M)$. Since Tor may be restricted to $TM_0 \approx M$, it follows that if Tor $\equiv 0$ and $p \in TM_0$, then

$$\mathrm{Tor}\,(X, Y)_p = (\nabla'_X Y)_p - (\nabla'_Y X)_p - [X, Y]_p\ ,$$

so that

$$(D_X Y)_p + \alpha(X, Y)_p - (D_Y X)_p - \alpha(Y, X)_p - [X, Y]_p = 0\ .$$

Thus we see that $\mathrm{Tor}_D\,(X, Y) = 0$ and $\alpha(X, Y) = \alpha(Y, X)$.

Definition. An $\mathfrak{F}'$-metric on TM is a map $G\colon \mathfrak{X}'(TM) \times \mathfrak{X}'(TM) \to \mathfrak{F}'(TM)$ which is C^∞, symmetric, positive definite, bilinear over $\mathfrak{F}'(TM)$, and has the additional property that if $A, B \in \mathfrak{X}'(TM)$ and $p \in TM_0$, then $G(A_p, B_p) = G(A, B)_p$.

We will say that ∇' is Riemannian with respect to the $\mathfrak{F}'$-metric G if on TM_0

$$(8)\qquad \mathrm{Tor}\,(A, B) = 0\ ,\qquad XG(C, E) = G(\nabla'_X C, E) + G(C, \nabla'_X E)\ ,$$

where $X \in \mathfrak{X}(M)$, $A, B, C, E \in \mathfrak{X}'(TM)$, and A, B are horizontal.

Theorem 6. *If ∇' is Riemannian with respect to an $\mathfrak{F}'$-metric having the property that horizontal and vertical vectors are orthogonal on TM_0, then D is Riemannian with respect to the induced metric in the horizontal bundle over $TM_0 \approx M$, ∇ is metric with respect to the induced metric in the vertical bundle, and $\alpha \equiv 0$.*

Proof. From the definition of an $\mathfrak{F}'$-metric it is clear that by restricting G to horizontal and vertical vector fields on $TM_0 \approx M$ we obtain metrics on the horizontal and vertical bundles over M. Suppose that $X, Y, Z \in \mathfrak{X}(M)$ with extensions A, B, C respectively to horizontal vector fields on a neighborhood of $p \in TM_0$. If ∇' is Riemannian with respect to the metric G, then on TM_0

$$AG(B, C) = G(\nabla'_A B, C) + G(B, \nabla'_A C)\ .$$

If $p \in TM_0$, then by Lemma 1 and the definition of an $\mathfrak{F}'$-metric we have

$$X_p G(Y, Z) = G(\nabla'_X Y, Z)_p + G(Y, \nabla'_X Z)_p \ .$$

Thus on $TM_0 \approx M$ we have

$$XG(Y, Z) = G(D_X Y + \alpha(X, Y), Z) + G(Y, D_X Z + \alpha(X, Z)) \ .$$

Also we see that if $\xi \in \mathfrak{X}^v(M)$, then $G(Y, \xi) = 0$ so that

$$XG(Y, \xi) = G(D_X Y + \alpha(X, Y), \xi) + G(Y, \nabla_X \xi) = G(\alpha(X, Y), \xi) = 0 \ ,$$

which implies that $\alpha \equiv 0$. Thus

$$XG(Y, Z) = G(D_X Y, Z) + G(Y, D_X Z)$$

on $TM_0 \approx M$. That $\mathrm{Tor}_D \equiv 0$ follows from Theorem 5. Finally, if $\xi, \eta \in \mathfrak{X}^v(M)$, then on $TM_0 \approx M$

$$XG(\xi, \eta) = G(\nabla'_X \xi, \eta) + G(\xi, \nabla'_X \eta) = G(\nabla_X \xi, \eta) + G(\xi, \nabla_X \eta) \ .$$

Suppose that the covariant derivative $D\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is Riemannian with respect to the metric g on M, and that the covariant derivative $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}^v(M) \to \mathfrak{X}^v(M)$ is metric with respect to the fiber metric h in the vertical bundle over M. If $X \in \mathfrak{X}(M)$, and A is a section of ${}^2_0\Pi\colon {}_2M \to M$, then we define the second order covariant derivative

$$\mathscr{D}_X A = D_X A^h + \nabla_X A^v$$

and the corresponding $\mathfrak{F}'$-derivative, using (3),

$$(9) \qquad (\nabla'_B A)_p = ({}^1_0\Pi_* \oplus \tilde{D})_p^{-1}(D_{{}^1_0\Pi_* B_p^h}\, {}^1_0\Pi_* A + \nabla_{{}^1_0\Pi_* B_p^h} \tilde{D}(A)) \ ,$$

If $A, B \in \mathfrak{X}'(TM)$, and we take

$$\langle A, B \rangle_p = g({}^1_0\Pi_* A_p, {}^1_0\Pi_* B_p) + h(\tilde{D}(A_p), \tilde{D}(B_p)) \ ,$$

then $\langle\ ,\ \rangle$ is an $\mathfrak{F}'$-metric.

Theorem 7. *∇' is Riemannian with respect to $\langle\ ,\ \rangle$.*

Proof. If $A, B \in \mathfrak{X}'(TM)$ and $X \in \mathfrak{X}(M)$, then we have on TM_0

$$\begin{aligned} X\langle A, B \rangle &= g(D_X\, {}^1_0\Pi_*\, A, {}^1_0\Pi_*\, B) + g({}^1_0\Pi_*\, A, D_X\, {}^1_0\Pi_*\, B) \\ &\quad + h(\nabla_X \tilde{D}(A), \tilde{D}(B)) + h(\tilde{D}(A), \nabla_X \tilde{D}(B)) \ , \end{aligned}$$

since D is Riemannian with respect to g, ∇ is metric with respect to h, and ${}^1_0\Pi_*(A \mid TM_0)$, $\tilde{D}(A \mid TM_0)$ are vector fields. Since

$$D_X\,{}^1_0\Pi_*A = {}^1_0\Pi_*({}^1_0\Pi_*\oplus\tilde{D})^{-1}(D_X\,{}^1_0\Pi_*A + \nabla_X\tilde{D}(A)) \quad \text{on} \quad TM_0$$

and a similar expression holds for ∇, we see that

$$X\langle A, B\rangle = \langle\nabla'_X A, B\rangle + \langle A, \nabla'_X B\rangle$$

on TM_0. If $A,B \in \mathfrak{X}'(TM)$ are horizontal and have the restrictions X and Y respectively to $TM_0 \approx M$, then for $p \in TM_0$

$$\mathrm{Tor}\,(A, B)_p = \mathrm{Tor}\,(X, Y)_p = (D_X Y - D_Y X - [X, Y])_p\ .$$

Thus Tor $\equiv 0$ on TM_0 since D is Riemannian.

If $A, B, C \in \mathfrak{X}'(TM)$, we define

(10) $$R(A, B)C = \nabla'_A\nabla'_B C - \nabla'_B\nabla'_A C - \nabla'_{[A,B]}C\ .$$

Theorem 8.

$$R(A, B)C = -R(B, A)C\ ,$$

and R is $\mathfrak{F}'(TM)$ multilinear.

Theorem 9. *If $A, B, C \in \mathfrak{X}'(TM)$, and A, B are horizontal, then for $p \in TM_0$*

$$R(A_p, B_p)C_p = (R(A, B)C)_p\ .$$

Proof. In terms of a coordinate chart at ${}^1_0\Pi(p)$ we have

$$A\,|\,TM_0 = a^{0i}X_i^h\,|\,TM_0\ , \qquad B\,|\,TM_0 = b^{0i}X_i^h\,|\,TM_0\ ,$$
$$C\,|\,TM_0 = C^{0i}X_i^h\,|\,TM_0 + C^{1i}X_i^v\,|\,TM_0\ ,$$

where a^{0i}, b^{0i}, C^{0i}, $C^{1i} \in \mathfrak{F}(M)$. Extend these to the vector fields

$$\bar{A} = a^{0i\prime}X_i^h\ , \quad \bar{B} = b^{0i\prime}X_i^h\ , \quad \bar{C} = C^{0i\prime}X_i^h + C^{1i\prime}X_i^v\ ,$$

where the accent denotes vertical lift. From Lemmas 1 and 2 and the definition of R we see that for $p \in TM_0$, R_p depends only upon the values of A, B, and C on TM_0. Consequently, we have on TM_0

$$\begin{aligned} R(A, B)C = R(\bar{A}, \bar{B})\bar{C} &= R(a^{0i\prime}X_i^h, b^{0j\prime}X_j^h)(C^{0k\prime}X_k^h + C^{1k\prime}X_k^v) \\ &= a^{0i\prime}b^{0j\prime}C^{0k\prime}R(X_i^h, X_j^h)X_k^h + a^{0i\prime}b^{0j\prime}c^{1k\prime}R(X_i^h, X_j^h)X_k^v\ . \end{aligned}$$

Thus we see that if A, B or C vanishes at a point $p \in TM_0$, then $(R(A, B)C)_p = 0$, and hence we may take

$$R(A_p, B_p)C_p = (R(A, B)C)_p\ .$$

Remark. This implies that R induces a tensor on $TM_0 \approx M$, since the restriction of $\mathfrak{F}'(TM)$ to $TM_0 \approx M$ may be identified with $\mathfrak{F}(M)$.

Using the fact that ∇' may be restricted to $TM_0 \approx M$ we have for $X, Y, Z \in \mathfrak{X}(M)$

$$\begin{aligned} \nabla'_X\nabla'_Y Z &= \nabla'_X(D_Y Z + \alpha(Y, Z)) = D_X D_Y Z + \nabla_X\alpha(Y, Z) + \alpha(X, D_Y Z) , \\ \nabla'_Y\nabla'_X Z &= D_Y D_X Z + \nabla_Y\alpha(X, Z) + \alpha(Y, D_X Z) , \\ \nabla'_{[X,Y]} Z &= D_{[X,Y]} Z + \alpha([X, Y], Z) . \end{aligned} \tag{11}$$

If D is torsion free, then

$$D_X Y - D_Y X = [X, Y] ,$$

so that

$$\nabla_{[X,Y]} Z = D_{[X,Y]} Z + \alpha(D_X Y, Z) - \alpha(D_Y X, Z) .$$

Using (11) we see that the horizontal component of R is

$$(R(X, Y)Z)^h = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z = \tilde{R}(X, Y)Z , \tag{12}$$

where $\tilde{R}$ is the curvature of the (first order) induced connection. (12) is analogous to the equation of Gauss. The vertical component of R is

$$\begin{aligned} (R(X, Y)Z)^v = \nabla_X\alpha(Y, Z) &+ \alpha(X, D_Y Z) - \alpha([X, Y], Z) \\ &- \nabla_Y\alpha(X, Z) - \alpha(Y, D_X Z) . \end{aligned}$$

Taking

$$\tilde{\nabla}_X\alpha(Y, Z) = \nabla_X\alpha(Y, Z) - \alpha(D_X Y, Z) - \alpha(Y, D_X Z)$$

we have in the case where D is torsion free

$$(R(X, Y)Z)^v = \tilde{\nabla}_X\alpha(Y, Z) - \tilde{\nabla}_Y\alpha(X, Z) , \tag{13}$$

which is formally the same as the equation of Codazzi. Finally, we have for $\xi \in \mathfrak{X}^v(M)$

$$\nabla'_X\nabla'_Y\xi = \nabla_X\nabla_Y\xi , \quad \nabla'_Y\nabla'_X\xi = \nabla_Y\nabla_X\xi , \quad \nabla'_{[X,Y]}\xi = \nabla_{[X,Y]}\xi ,$$

and hence

$$R(X, Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi = \bar{R}(X, Y)\xi . \tag{14}$$

We call $\bar{R}$ the "vertical curvature tensor" of M.

If $\langle\ ,\ \rangle$ is an $\mathfrak{F}'$-metric on TM, and ∇' is Riemannian with respect to $\langle\ ,\ \rangle$, then we define, for $A, B, C, D \in \mathfrak{X}'(TM)$,

$$R(A, B, C, D) = \langle A, R(C, D)B \rangle . \tag{15}$$

Theorem 10. *If $A, B \in \mathfrak{X}'(TM)$ and $X, Y \in \mathfrak{X}(M)$, then*

$$R(A, B, X, Y) = -R(A, B, Y, X) , \qquad R(A, B, X, Y) = -R(B, A, X, Y) .$$

Proof. The first of these follows from the skew-symmetry of $R(X, Y)B$, and the second from the fact that, since $\langle\ ,\ \rangle$ is Riemannian,

$$XY\langle A, B\rangle = \langle \nabla'_X \nabla'_Y A, B\rangle + \langle \nabla'_Y A, \nabla'_X B\rangle + \langle \nabla'_X A, \nabla'_Y B\rangle + \langle A, \nabla'_X \nabla'_Y B\rangle ,$$
$$[X, Y]\langle A, B\rangle = \langle \nabla'_{[X,Y]} A, B\rangle + \langle A, \nabla'_{[X,Y]} B\rangle$$

on $TM_0 \approx M$. Hence

$$XY - YX - [X, Y] = 0$$
$$= \langle R(X, Y)A, B\rangle + \langle A, R(X, Y)B\rangle .$$

Let $G(A, B) = \langle A, A\rangle\langle B, B\rangle - \langle A, B\rangle^2$, and

$$K(A, B) = R(A, B, A^h, B^h)/G(A, B) . \tag{16}$$

Theorem 11. *If $p \in TM_0 \approx M$ and $A, B \in {}^2M_p$, then the scalar $K(A, B)$ depends only upon the hyperplane of ${}^2M_p$ spanned by A and B.*

Proof. We see that

$$K(A, B) = K(B, A) = K(rA, sB) = K(A + tB, B) .$$

Thus if $ad - cb \neq 0$, then

$$K(A, B) = K(aA + bB, cA + dB) .$$

Corollary. *If $\alpha \equiv 0$ and $A, B \in \mathfrak{X}'(TM)$ are horizontal with $A \mid TM_0 = X$, $B \mid TM_0 = Y$, then*

$$K(A, B) = \tilde{K}(XY) ,$$

where $\tilde{K}$ is the curvature of the induced (first order) connection on M.

If $X \in \mathfrak{X}(M)$, let $X^* \in \mathfrak{X}^v(M)$ denote the vertical vector field having the property that $D(X^*) = X$. Then to complement the first order or horizontal curvature $\tilde{K}$ we define the vertical curvature

$$\bar{K}(X, Y) = R(X^*, Y^*, X, Y)/G(X, Y) . \tag{17}$$

References

[1] R. H. Bowman, *Second order connections,* J. Differential Geometry **7** (1972).
[2] S. Kobayashi & K. Nomizu, *Foundations of differential geometry,* Vols. I and II, Interscience, New York, 1963 and 1969.

ARKANSAS STATE UNIVERSITY